Minimal-Time System Design Algorithm Prediction by means of Lyapunov Function Analysis

ALEXANDER ZEMLIAK
Department of Physics and Mathematics
Puebla Autonomous University
Av. San Claudio y Rio Verde, Puebla, 72570
MEXICO

Abstract: – The minimal time system design algorithm was defined as the problem of functional minimization of the control theory. By this methodology the aim of the system design process with minimal computer time is presented as a transition process of dynamic system with minimal transition time. The optimal sequence of the control vector switch points was determined as a principal characteristic of the minimal-time system design algorithm. The different forms of the Lyapunov function were proposed to analyze the behavior of a design process. The special function that is a combination of Lyapunov function and its time derivative was proposed to predict the optimal control vector structure to construct a minimal-time system design algorithm.

Key Words: - Minimal-time system design, control theory application, Lyapunov function.

1 Introduction

The problem of the computer time reduction of a large system design is one of the essential problems of the total quality design improvement. Besides the traditionally used ideas of sparse matrix techniques and decomposition techniques [1]-[5] some another ways were proposed to reduce the total computer design time [6]-[7]. The generalized approach for the analog system design on the basis of the control theory formulation was elaborated in some previous works, for example [8]. This approach serves for the minimal-time design algorithm definition. On the other hand this approach gives the possibility to analyze with a great clearness the design process while moving along the trajectory curve into the design space. The main conception of this theory is the introduction of the special control functions, which, on the one hand generalize the design process and, on the other hand, they give the possibility to control the design process to achieve the optimum of the design cost function for the minimal computer time. This possibility appears because practically an infinite number of the different design strategies that exist within the bounds of the theory. The different design strategies have the different operation number and executed computer time. On the bounds of this conception, the traditional design strategy is only a one representative of the enormous set of different design strategies. As shown in [8] the potential computer time gain that can be obtained by the new design problem formulation increases when the size and complexity of the system increase. However it is realized only in case when the algorithm for the optimal design strategy is constructed.

We can define the formulation of the intrinsic properties and special restrictions of the optimal design strategy as one of the first problems that needs to be solved for the optimal algorithm construction.

2 Problem Formulation

The design process for any analog system design can be defined in discrete form [8] as the problem of the generalized cost function $F(X,U)$ minimization by means of the vector equation (1) with the constraints (2):

$$X^{s+1} = X^s + t_s \cdot H^s \qquad (1)$$

$$\left(1-u_j\right) g_j(X) = 0, \qquad j = 1,2,\ldots,M \qquad (2)$$

where $X \in R^N$, $X=(X',X'')$, $X' \in R^K$ is the vector of the independent variables and the vector $X'' \in R^M$ is the vector of dependent variables ($N=K+M$), $g_j(X)$ for all j presents the system model, s is the iterations number, t_s is the iteration parameter, $t_s \in R^1$, $H \equiv H(X,U)$ is the direction of the generalized cost function $F(X,U)$ decreasing, U is the vector of the special control functions

$U=(u_1,u_2,\ldots,u_m)$, where $u_j\in\Omega$; $\Omega=\{0;1\}$. The generalized cost function $F(X,U)$ is defined as:

$$F(X,U)=C(X)+\psi(X,U) \qquad (3)$$

where $C(X)$ is the non negative cost function of the design process, and $\psi(X,U)$ is the additional penalty function:

$$\psi(X,U)=\frac{1}{\varepsilon}\sum_{j=1}^{M}u_j\cdot g_j^2(X) \qquad (4)$$

This formulation of the problem permits to redistribute the computer time expense between the solution of problem (2) and the optimization procedure (1) for the function $F(X,U)$. The control vector U is the main tool for the redistribution process in this case. Practically an infinite number of the different design strategies are produced because the vector U depends on the optimization procedure current step. The problem of the optimal design strategy search is formulated now as the typical problem for the functional minimization of the control theory. The functional that needs to minimize is the total CPU time T of the design process. This functional depends directly on the operations number and on the design strategy that has been realized. The main difficulty of this definition is unknown optimal dependencies of all control functions u_j.

The continuous form of the problem definition is more adequate for the control theory application. This continuous form replaces Eq. (1) and can be defined by the next formula:

$$\frac{dx_i}{dt}=f_i(X,U), \qquad i=0,1,\ldots,N \qquad (5)$$

This system together with equations (2), (3) and (4) composes the continuous form of the design process. The structural basis of different design strategies that correspond to the fixed control vector includes 2^M design strategies. The functions of the right hand part of the system (5) are determined for example for the gradient method as:

$$f_i(X,U)=-\frac{\delta}{\delta x_i}F(X,U) \qquad i=1,2,\ldots,K \qquad (6)$$

$$f_i(X,U)=-u_{i-K}\frac{\delta}{\delta x_i}F(X,U)+\frac{(1-u_{i-K})}{t_s}\left\{-x_i^s+\eta_i(X)\right\}$$
$$i=K+1,K+2,\ldots,N \qquad (6')$$

where the operator $\frac{\delta}{\delta x_i}$ hear and below means $\frac{\delta}{\delta x_i}\varphi(X)=\frac{\partial\varphi(X)}{\partial x_i}+\sum_{p=K+1}^{K+M}\frac{\partial\varphi(X)}{\partial x_p}\frac{\partial x_p}{\partial x_i}$, x_i^s is equal to $x_i(t-dt)$; $\eta_i(X)$ is the implicit function ($x_i=\eta_i(X)$) that is determined by the system (2).

The control variables u_j have the time dependency in general case. The equation number j is removed from (2) and the dependent variable x_{K+j} is transformed to the independent when u_j=1. This independent parameter is defined by the formulas (5), (6'). In this case there is no difference between formulas (6) and (6'). On the other hand, the equation (5) with the right part (6') is transformed to the identity $\frac{dx_i}{dt}=\frac{dx_i}{dt}$, when u_j=0, because. $\eta_i(X)-x_i^s=x_i(t)-x_i(t-dt)=dx_i$. It means that at this time moment the parameter x_i is dependent one and the current value of this parameter can be obtained from the system (2) directly. This transformation of the vectors X' and X'' can be done at any time moment. The function $f_0(X,U)$ is determined as the necessary time for one step of the system (5) integration. This function depends on the concrete design strategy. The additional variable x_0 is determined as the total computer time T for the system design. In this case we determine the problem of the time-optimal system design as the classical problem of the functional minimization of the optimal control theory. In this context the aim of the optimal control is to result each function $f_i(X,U)$ to zero for the final time T, to minimize the cost function and the total computer time x_0.

It is necessary to find the optimal behavior of the control functions u_j during the design process to minimize the total design computer time. The functions $f_i(X,U)$ are piecewise continued as the temporal functions.

The idea of the system design problem formulation as the functional minimization problem of the control theory is not depend of the optimization method and can be embedded into any optimization procedures. In this paper the gradient method is used, nevertheless any optimization method can be used as shown in [8].

Now the analog system design process is formulated as a dynamic controllable system. The time-optimal design process can be defined as the dynamic system with the minimal transition time in this case. So we need to find the special conditions to minimize the transition time for this dynamic system.

3 Lyapunov Function Definitions

On the basis of the analysis in previous section we can conclude that the minimal-time algorithm has one or some switch points in control vector where the switching is realize among different design strategies. As shown in [9] it is necessary to switch the control vector from like modified traditional design strategy to like traditional design strategy with an additional adjusting. Some principal features of the time-optimal algorithm were determined previously. These are: 1) an additional acceleration effect that appeared under special circumstances [9]; 2) the start point special selection outside the separate hyper-surface to guarantee the acceleration effect, at least one negative component of the start value of the vector *X* is can be recommended for this; 3) an optimal structure of the control vector with the necessary switch points. The two first problems were discussed in [9-10]. The third problem is discussed in the present paper.

The main problem of the time-optimal algorithm construction is unknown optimal sequence of the switch points during the design process. We need to define a special criterion that permits to realize the optimal or quasi-optimal algorithm by means of the optimal switch points searching. A Lyapunov function of dynamic system serves as a very informative object to any system analysis in the control theory. We propose to use a Lyapunov function of the design process for the optimal algorithm structure revelation, in particular for the optimal switch points searching.

There is a freedom of the Lyapunov function choice because of a non-unique form of this function. Let us define the Lyapunov function of the design process (2)-(6) by the following expression:

$$V(X) = \sum_i (x_i - a_i)^2 \qquad (7)$$

where a_i is the stationary value of the coordinate x_i, in other words the set of all the coefficients a_i is the main objective of the design process. The function (7) satisfies all of the conditions of the standard Lyapunov function definition for the variables $y_i = x_i - a_i$. In fact the function $V(Y) = \sum_i y_i^2$ is the piecewise continue. Besides there are three characteristics of this function: i) *V(Y)*>0, ii) *V(0)*=0, and iii) $V(Y) \to \infty$ when $\|Y\| \to \infty$. Inconvenience of the formula (7) is an unknown point $a=(a_1, a_2, ..., a_N)$, because this point can be reached at the end of the design process only. We can use this form of the Lyapunov function if we already found the design solution someway. On the other hand, it is very important to control the stability of the design process during the optimization procedure. In this case we need to construct other form of the Lyapunov function that doesn't depend on the unknown stationary point. Let us define two new forms of the Lyapunov function by the next formulas:

$$V(X,U) = [F(X,U)]^r \qquad (8)$$

$$V(X,U) = \sum_i \left(\frac{\partial F(X,U)}{\partial x_i} \right)^2 \qquad (9)$$

where *F(X,U)* is the generalized cost function of the design process. The formula (8) can be used when the general cost function is non negative and has zero value at the stationary point *a*. Other formula can be used always because all derivatives $\partial F / \partial x_i$ are equal to zero in the stationary point *a*. So, the function *V* for both formulas has properties: *V(a,U)=0*, *V(X,U)>0* for all *X* and at last, this function increases in a sufficient large neighborhood of the stationary point. Besides, the function *V* is the function of the vector *U* too, because all coordinates x_i are the functions of the control vector *U*.

We can define now the design process as a transition process for controllable dynamic system that can provide the stationary point (optimal point of the design procedure) during some time. The problem of the time-optimal design algorithm construction can be formulated now as the problem of the transition process searching with the minimal transition time. There is a well-known idea [11]-[13] to minimize the time of the transition process by means of the special choice of the right hand part of the principal system of equations, in our case these are the functions $f_i(X,U)$. It is necessary to change the functions $f_i(X,U)$ by means of the control vector *U* selection to obtain the maximum speed of the Lyapunov function decreasing (the maximum absolute value of the Lyapunov function time derivative $\dot{V} = dV/dt$). Normally the time derivative of the Lyapunov function is non positive for the stable processes. However we can define now more informative function as a time derivative of Lyapunov function relatively the Lyapunov function: $W = \dot{V}/V$. In this case we can compare the different design strategies by means of the function *W(t)* behavior and we can search the optimal position for the control vector switch points.

4 Structural Basis Analysis

All examples were analyzed for the continuous form of the optimization procedure (5). Functions *V(t)* and *W(t)* were the main objects of the analysis and its behavior has been analyzed for all strategies that compose the structural basis of the design general methodology. The behavior of the functions *V(t)* and *W(t)* for the network of Fig. 1 is shown in Fig. 2a, and Fig. 2b.

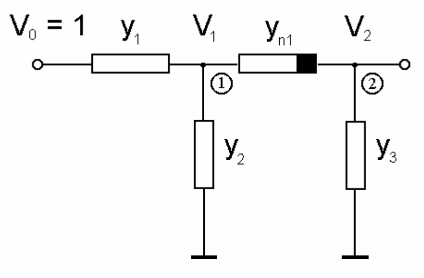
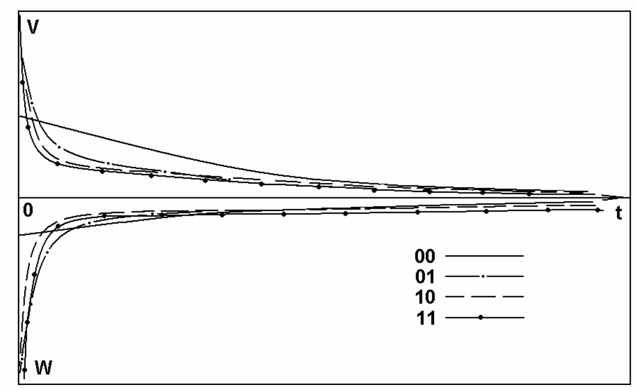


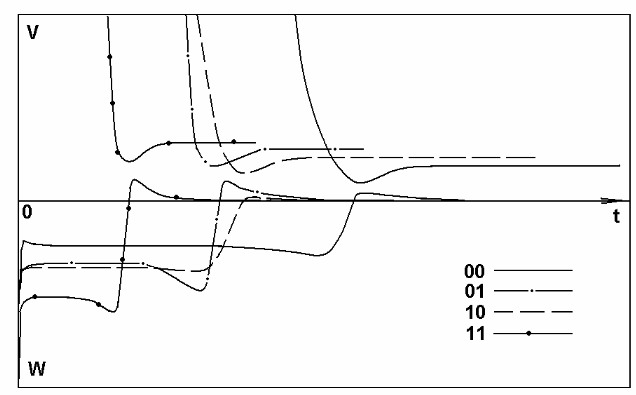
Figure 1. Two-node nonlinear passive network.

The nonlinear element has the following dependency: $y_{n1} = y_0 + b(V_1 - V_2)^2$. The vector X includes five components: $x_1^2 = y_1$, $x_2^2 = y_2$, $x_3^2 = y_3$, $x_4 = V_1$, $x_5 = V_2$. The model of this network (2) includes two equations (M=2) and the optimization procedure (5) includes five equations.

The network in Fig. 1 is characterized by two dependent parameters (two nodal voltages) and the control vector includes two control functions: $U=(u_1, u_2)$. The structural basis of design strategies includes four design strategies; 00, 01, 10, 11. The Lyapunov function was calculated by formula (8) for r=0.5. As we can see from Fig. 2 the functions *V(t)* and *W(t)* can give an exhaustive explanation for the design process characteristics. Fig. 2a shows these functions behavior for the initial part of the design process (2% of the total design time). First of all we can conclude that the speed of decreasing of the Lyapunov function is inversely proportional to the design time. The minimal value of the Lyapunov function that corresponds to the maximum precision is approximately equal for all strategies and exactly is equal to $8.7_{10}\text{-}6$, $1.7_{10}\text{-}5$, $1.3_{10}\text{-}5$, $2.0_{10}\text{-}5$ for the strategies 00, 01, 10, 11 accordingly. We can see from Fig. 2b that after the minimal value decision the Lyapunov function increases a little. This small increasing corresponds to the small positive value of the Lyapunov function time derivative. Later on this derivative aspire to zero and the Lyapunov function has a permanent value.



(a)



(b)

Figure 2. Behavior of the functions *V(t)* and *W(t)* for four design strategies during the design process for network in Fig.1; (a) – initial part of the design process, (b) – design process the whole with the final part in detail.

The relative design time for four design strategies is equal to 1, 0.44, 0.78 and 0.3 for the strategies 00, 01, 10, 11 accordingly. This time was defined for the time point with the minimal value of the function *V*. As we can see from Fig. 2b a large absolute value of the function *W(t)* corresponds to a more rapid decreasing of the function *V(t)* and a smaller computer design time.

Other example corresponds to the network in Fig.3. The vector X includes six components: $x_1^2 = y_1$, $x_2^2 = y_2$, $x_3^2 = y_3$, $x_4 = V_1$, $x_5 = V_2$, $x_6 = V_6$.

The model of this network (2) includes three equations (M=3) and the optimization procedure (5) includes six equations. The total structural basis contains eight different strategies. The control vector has three components in this case and the structural basis consists of eight design strategies. The control vector includes three control functions: $U=(u_1, u_2, u_3)$.

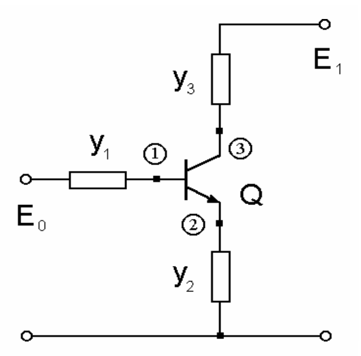
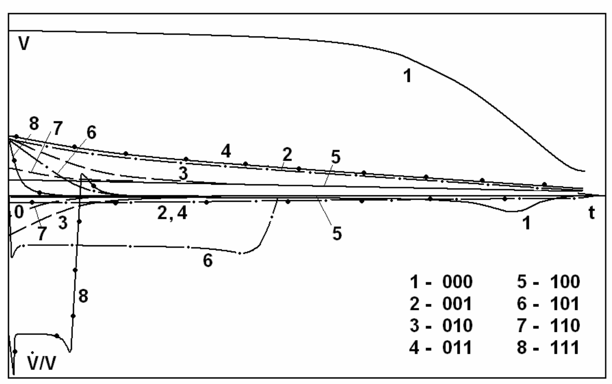
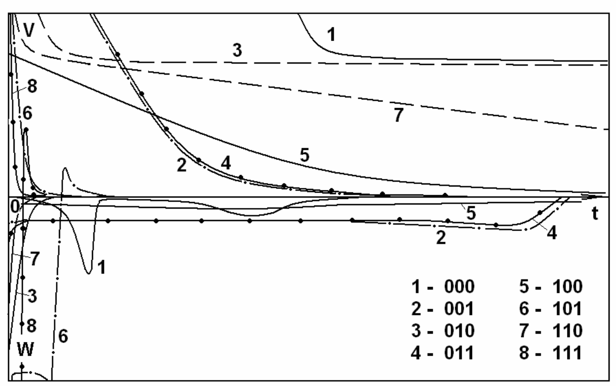


Figure 3. Three-node nonlinear active network.



(a)



(b)

Figure 4. Behavior of the functions *V(t)* and *W(t)* for different design strategies during the design process for network in Fig.3; (a) – initial part of the design process, (b) – design process the whole with the final part in detail.

As for the first example, Fig. 4a shows the behavior of the functions *V(t)* and *W(t)* for the initial part of the design process. The graphs in Fig. 4b correspond to a time interval when the majority of the design strategies are finished. The strategies with control vector 101 and 111 have extremely large value of the relative derivative *W* from the beginning of the design process and that is why the Lyapunov function is decreases very rapidly. The relative design time is very small for two these strategies and it is equal to 0.00057 and 0.00018 accordingly. The strategies with the control vector 001, 011 and 100 have the sufficient level of the function *W* during the analyzed interval and the relative design time is equal to 0.0054, 0.0061 and 0.0114 accordingly. Nevertheless three other design strategies with the control vector 000, 010 and 110 are not finished during the presented interval. It occurs because the function *W* for these strategies decreases rapidly while the Lyapunov function had a relatively large value. After this the Lyapunov function decreases very slowly and the relative design time is equal to 1.0, 0.127 and 0.027 accordingly. So, the main feature of the analyzed examples can be formulated by the next manner: the behavior of the Lyapunov function *V* and the relative time derivative *W* with confidence determine the design time. It means that it is possible be guided by means of these functions to predict the computer design time for any design strategy. We could analyzed the functions *V(t)* and *W(t)* behavior for the initial time interval only for the different strategies and on the basis of this analysis we can predict the strategies that have a minimal computer design time.

5 Optimal Strategy Prediction

As discussed above the principal element of the minimal time design algorithm is the optimal position of the control vector switch point. Some networks were analyzed from this viewpoint. The results of the analysis for the network in Fig.2 are shown in Fig 5 and Table 1.

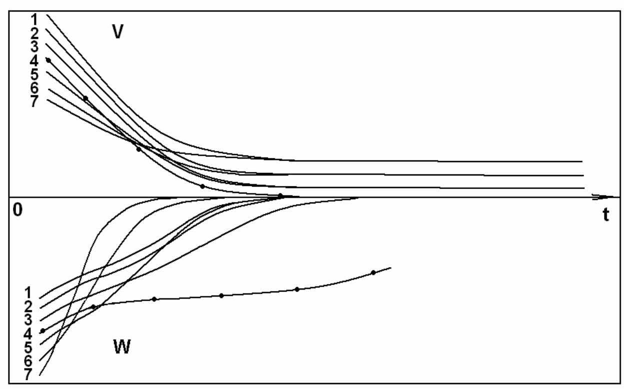


Figure 5. Behavior of the functions *V(t)* and *W(t)* during the design process after the control vector switch for seven consecutive steps of the switch points (from 33 to 39) for network in Fig.2.

Behavior of the functions *V(t)* and *W(t)* help us to determine the optimal position of the control vector switch point. Take into account the preliminary reasons about the optimal algorithm structure [9] we have been analyzed the strategy of the special structure that consists of two parts. The first part is defined by the control vector (111) and the second part is defined by the control vector (000). The optimal switch point was a principal objective of this analysis. The consecutive change of the switch point was realized for the integration steps from 2 to 50. The behavior of the functions *V(t)* and *W(t)* for the optimal switch step and some steps near the optimal are shown in Fig. 5. The data which correspond of these graphs are presented in Table 1. The analysis shows that the optimal switch point corresponds to the step 36 (graph with dots). The computer design time has a minimal value for this step.

Table 1.

N	Switch point	Iterations number	Total design time (sec)
1	33	2433	0.404
2	34	2180	0.361
3	35	1748	0.289
4	36	61	0.01
5	37	1705	0.281
6	38	2111	0.349
7	39	2349	0.389

We can see that the function *W(t)* has a maximum absolute value for the optimal switch step (number 4) leading off the 15th integration step. It means that from the 15th step we can confidently predict the optimal switch point position that leads to the minimal computer design time. So, the structure of the optimal control vector i.e. the structure of the time optimal design strategy can be defined by means of the function *W(t)* analysis.

6 Conclusion

The problem of the minimal-time design algorithm construction can be solved adequately on the basis of the control theory. The design process in this case is formulated as the controllable dynamic system. The Lyapunov function and its time derivative include the sufficient information to select more perspective design strategies from infinite set of the different design strategies that exist into the general design methodology. The special function *W(t)* was proposed to predict the structure of the time optimal design strategy. This function can be used also to construct the optimal sequence of the control vector switch points. The solution of this problem permits to construct the minimal-time system design algorithm.

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